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LETTER TO THE EDITOR

Lie symmetries versus integrability in evolution equations

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Abstract. Motivated by some recent results obtained concerning the invariance properties of a particular family of generalized KdV equations, we investigate the possible significance of classical Lie invariance groups as a test for complete integrability.

The great amount of success attained in the study of completely integrable (CI) evolution equations (EC) of the form

$$u_t = F(x, t, u, u_1, u_2, \ldots, u_N) \qquad N \geqslant 2 \tag{1}$$

where $u = u(x, t)$ and $u_j = \partial^j u(x, t)/\partial x^j$, can disguise the astonishing fact that the mutual dependence of the genuine properties held by CICC is not yet properly understood.

As a matter of fact, this is the case for two of them which have been used as tests for complete integrability, in an attempt to isolate and classify CIEE: (Ex_{LB}) the existence of an infinite number of non-trivial Lie–Bäcklund symmetries; and (Ex_{CD}) the existence of an infinite number of non-trivial conserved densities $\rho(x, t, u, u_1, u_2, \ldots, u_m)$. In practice, the tests have often been based on a weaker hypothesis: the existence of some Lie–Bäcklund symmetry or conservation law of high order (order 5 for generalized KdV equations, for instance, see [2]).

Remark. It is worth mentioning that, to our knowledge, no example of (Ex_{CD}) with m bounded has been found. It remains as an interesting open problem to decide if such a case is really impossible. In other words, does every CICE necessarily have a sequence of non-trivial conserved densities $\rho(x, t, u, u_1, u_2, \ldots, u_m)$ with $m \to \infty$?

It is well known [3] that (Ex_{LB}) does not imply (Ex_{CD}) for even N: the heat equation $u_t = u_{xx}$ provides a counter-example. Furthermore, it was pointed out in [4] that even for odd N the existence of high-order Lie–Bäcklund symmetries can still be 'accidental'; in fact, the equation $u_t = u_3 + u^2 u_2 + 3uu_1^2 + \frac{1}{3}u^4 u_1$ admits a genuine high-order Lie–Bäcklund symmetry

$$u_5 + \tfrac{5}{3}u^2 u_4 + \tfrac{10}{9}(12uu_1 + u^4)u_3 + \tfrac{25}{3}uu_2^2 + \tfrac{10}{27}(45u_1^2 + 36u^3 u_1 + u^6)u_2 + \tfrac{5}{81}(252u_1^2 + 42u^3 u_1 + u^6)u^2 u_1$$

whereas its only conserved density is $\rho = u^2$.

Pessimistic as the proceding remarks may seem, (Ex_{LB}) has proved to be, by itself or in conjunction with (Ex_{CD}), a very reliable guide in the classification of CIEE (see [5] for a detailed review on this subject).

In the meantime, the modest classical Lie symmetries (CLS) have been used in different contexts: separation of variables, similarity solutions, and so on. However, their ability to isolate CIEE seems to be an open question. Now, since CLS are finite in number and easy to find, they would provide a suitable test for complete integrability.

Two recently discovered facts provide support to the idea that having an exceptionally large group G of CLS might be a characteristic feature of CIEE. First of all, Gazeau and Winternitz [1] have carried out an exhaustive analysis of the groups G of CLS for the family of generalized KdV equations

$$u_t = g(x, t)u_3 + h(x, t)uu_1 . \tag{2}$$

They arrange them into classes according to $\dim(G)$, the maximal value of $\dim(G)$ being 4. The linear equation $u_t = u_3$ ($g = 1$, $h = 0$) belongs to this maximal 4-class. As for nonlinear equations, a most remarkable fact arises: a nonlinear equation (2) is contained in the 4-class if and only if it is equivalent (up to point transformations) to the KdV equation, $g = h = 1$.

On the other hand, Sokolov [6] has found some EE which have larger symmetry groups than their linear counterparts. It is not known if they are CI, but there is some evidence in this direction. Anyway, none of them is linear in u_N.

In view of the preceding results, one is tempted to formulate the following naive conjecture:

(C) For a member Ω in the family of odd-order equations of the form $u_t = g(x, t, u, u_1, \ldots, u_{N-1})u_N + h(x, t, u, u_1, \ldots, u_{N-1})$, linear in u_N, the following statements are equivalent:

(a) Ω is (equivalent to) a CI equation,

(b) Ω belongs to the maximal class.

For the sake of simplicity, it seems convenient to test (C) on some third-order family of equations such that:

(i) it contains the linear equation as a limit case; and

(ii) the CI members of the family are known.

Both requirements are satisfied by

$$u_t = g(t, u, u_1)u_3 . \tag{3}$$

The CI members of (3) are, up to simple change of variables (see [7]):

$$g(u, u_1) - (\alpha u^2 + \beta u + \gamma)^{3/2} \tag{4}$$

and

$$g(u, u_1) = (u_1 + k)^3 . \tag{5}$$

The Harry–Dym equation corresponds to the choice $\alpha = 1$, $\beta = \gamma = 0$ in (4).

For the moment, let us restrict ourselves to the subfamily $u_t = g(u)u_3$. It is easy to check that there are three classes for $u_t = g(u)u_3$:

(Cl_5) $\quad g(u) = u^3$.

Its invariance group consists of two translations ∂_x, ∂_t, two dilatations $x\partial_x + u\partial_u$, $3t\partial_t - u\partial_u$, and a fifth operator $\sigma = x^2\partial_x + 2xu\partial_u$. It is this σ which makes u^3 different from the other powers $g(u) = u^p$, $p \neq 3$.

(Cl_4) $\quad g(u) = u^p, p \neq 3 \quad$ and $\quad g(u) = \mathrm{e}^u$.

All groups in this class contain 2 translations and 1 dilatation. The fourth generator is another dilatation for $g(u)=u^p$, $p\neq 3$ and $\partial_u - t\partial_t$ for $g(u)=e^u$.

(Cl$_3$) All other $g(u)$. This is the generic class. The minimal group for $u_t=g(u)u_3$ includes x and t translations, and a g-independent dilatation.

Hence, we conclude that statement (C) does not hold. While $u_t=u^3u_3$ (maximal class) is CI, there are also CI equations in the (non-maximal) 4-class (e.g. $u_t=u^{3/2}u_3$) and even in the minimal 3-class (e.g. equation (4) with generic α, β, γ).

Nevertheless, one is still tempted to make the ultimate effort in order to save (C) to some extent, so let us try a weaker conjecture.

(C*) If a member Ω in the family of odd-order equations of the form $u_t=g(x, t, u, u_1, \ldots, u_{N-1})u_N+h(x, t, u, u_1, \ldots, u_{N-1})$, linear in u_N, is contained in the maximal class, then it is equivalent to a CI equation.

Even in this alternative form, it would provide a very convenient way to isolate CIEE. Unfortunately, an inspection of the subfamily $u_t=g(u_1)u_3$ will be enough to invalidate (C*). Indeed, there are only two classes for $u_t=g(u_1)u_3$:

(Cl$_5$) $\qquad g(u_1)=u_1^p,\ p\neq 0 \qquad$ and $\qquad g(u_1)=e^{u_1}$.

(Cl$_4$) All other $g(u_1)$. In this case, the invariance group of every equation $u_t=g(u_1)u_3$ contains x, t and u-translations, and a g-independent dilatation given by $x\partial_x+3t\partial_t+u\partial_u$.

Since not all equations in the maximal class are CI, we conclude that (C*) does not hold. Even more is true. Since the maximal class (Cl$_5$) of $u_t-g(u_1)u_3$ does contain equations which are not CI, and since $5>\Sigma_3$, conjecture (C*) does not hold even if one adds to it the requirement that the maximal class has dimension greater than Σ_N.

Although we are led to the disappointing conclusion that an exceptionally large group of CLS does not provide any valuable clue for complete integrability, a final question naturally arises. Does the fact that the invariance group contains a number of generators bigger than Σ_N involve some degree of linearizability?

In order to clarify this question, let us recall a related result. The invariance group G of any linear differential equation $L[x, t, u]=0$ consists of two pieces: (i) a finite-dimensional Lie group, each of its infinitesimal generators being associated to a parameter; and (ii) an infinite-dimensional set of symmetries $W(x, t)\partial_u$, where $W(x, t)$ denotes an arbitrary solution of $L[x, t, W]=0$. It was proved in [8] that the appearance of such a 'free' function in G characterizes linearizable equations [8].

Now, it is far from easy to grasp what kind of partial linearizability could be associated to the existence of an extra parameter, instead of an extra free function, in the invariance group. Equation $u_t=u^3u_3$ constitutes a convenient touchstone, owing to the 'extra' symmetry $\sigma=x^2\partial_x+2xu\partial_u$, associated to the local group action [9]

$$(x, t, u)\rightarrow\left(\frac{x}{1-\varepsilon x}, t, \frac{u}{1-2\varepsilon x}\right).$$

A possible way to handle this question could be the so-called factorization method from [10]. In fact, equations $u_t=u^2u_2$ and $u_t=u_2$ are related in this way. However, the situation seems to be very different for the couple $u_t=u^3u_3$ and $u_t=u_3$, since the most obvious symmetries do not lead to the desired link. Perhaps a more detailed analysis based on optimal systems of subalgebras would help to understand this intriguing feature.

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